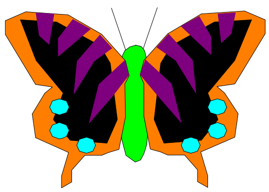
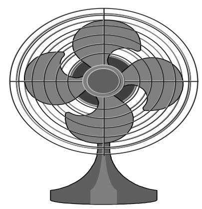
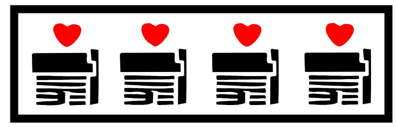
Kaleidoscopes, Hubcaps, and Mirrors
Problem 1.1 Notes

When part of an object or design is repeated to create a balanced pattern, we say that the object or design has symmetry.



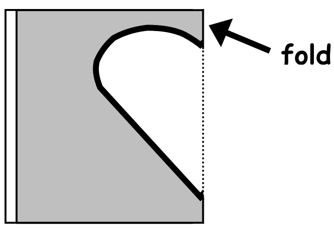


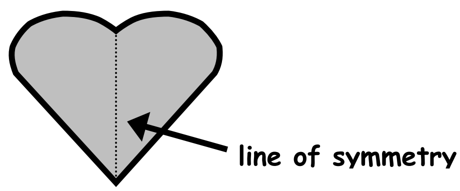


You can find examples of symmetry all around you. Artists use symmetry to make designs that are pleasing to the eye. Architects use symmetry to create a sense of balance in their buildings.

There are several types of symmetry:

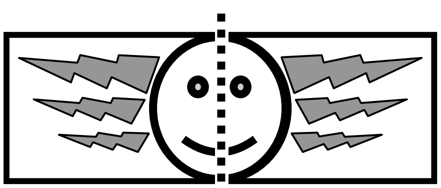
- the butterfly is an example of **<u>Reflectional</u>** symmetry (also called line symmetry or mirror symmetry)
- the fan blade is an example of **<u>Rotational</u>** symmetry
- the wall paper border is an example of **<u>Translational</u>** symmetry



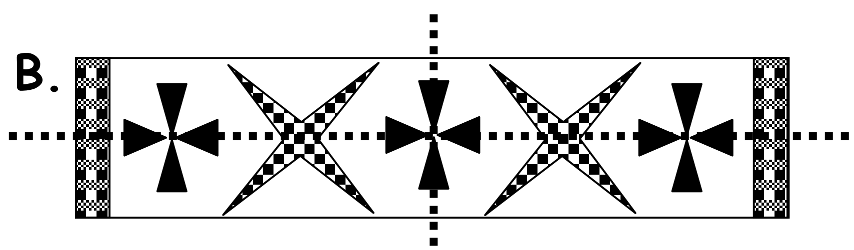


You can produce a symmetric design by folding a sheet of paper in half and making cuts. The fold through the center of the heart is the line of symmetry. A line of symmetry divides a design into halves that are mirror images. If you fold a design on a line of symmetry, the halves will exactly match. Mirrors and tracing paper are useful for checking reflectional symmetry.

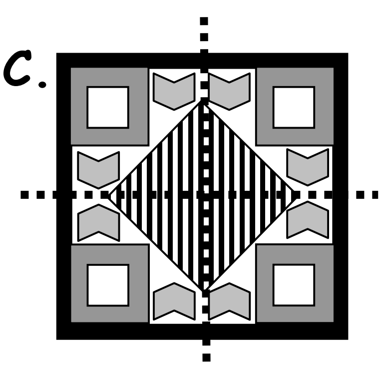
A.



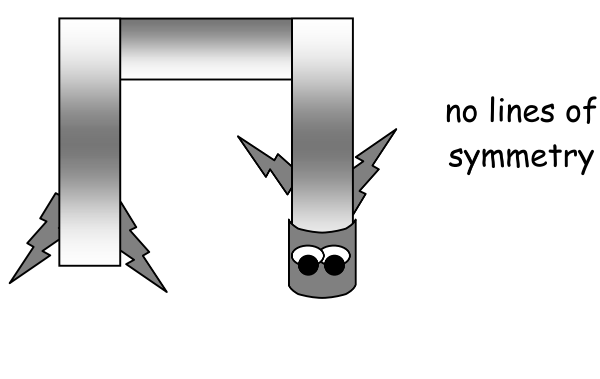
B.



C.



D.



Kaleidoscopes, Hubcaps, and Mirrors
Problem 1.2 Notes

Some designs have <u>rotational symmetry,</u> which means that the figure can be rotated about its centerpoint to a position in which it looks the same as the original design. The <u>angle of rotation</u> is the smallest angle through which a design can be rotated to coincide with the original design.

Rotational symmetry can be found in many objects that rotate about a centerpoint. For example the automobile hubcaps shown have rotational symmetry. These hubcaps are also on Labsheet 1.2.

A. Determine the angle of rotation for each hubcap. Explain how you found the angle.

Hubcap 1: 72°
360° ÷ 5 equal sections = 72°

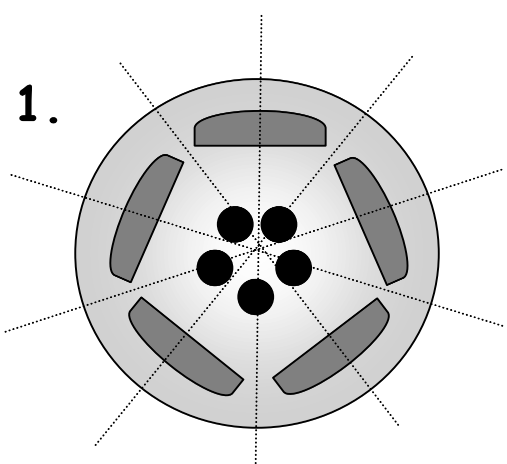
Hubcap 2: 120°
360° ÷ 3 equal sections = 120°

Hubcap 3: 36°
360° ÷ 10 equal sections = 36°

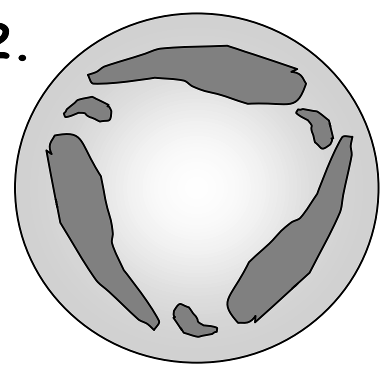
Hubcap 4: 40°
360° ÷ 9 equal sections = 40°

B. Some of the hubcaps also have reflectional symmetry. Sketch all the lines of e symmetry for each hubcap.

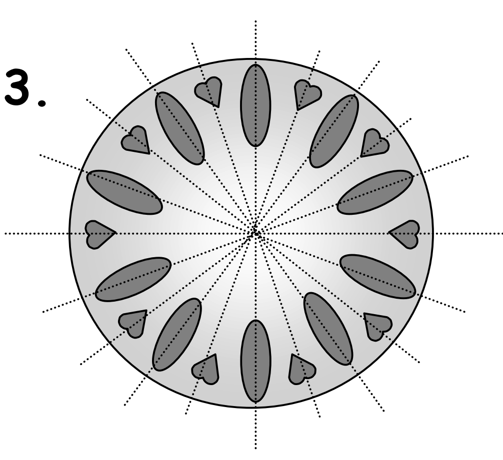
1.



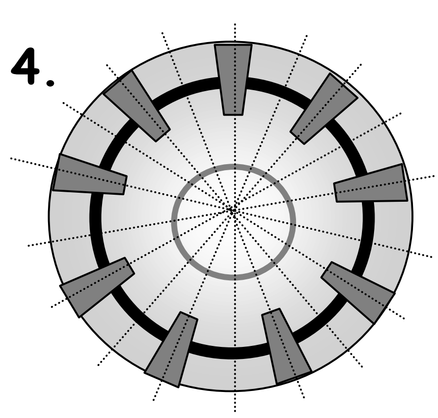
2.



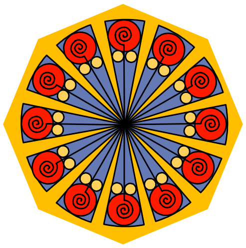
3.



4.



Kaleidoscopes, Hubcaps, and Mirrors
Problem 1.3 Notes

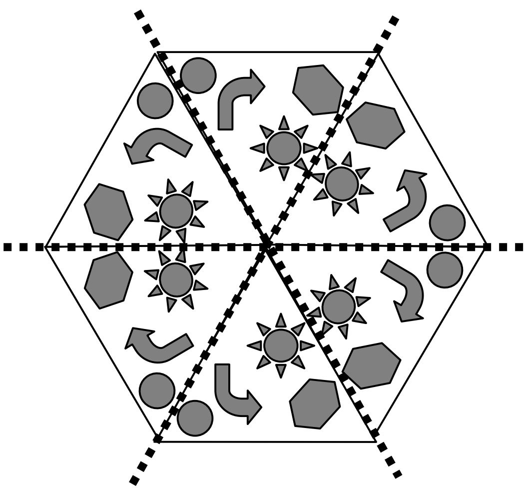


A kaleidoscope is a tube containing colored beads or pieces of glass and carefully placed mirrors. When a kaleidoscope is held to the eye and rotated, the viewer sees colorful symmetric patterns.

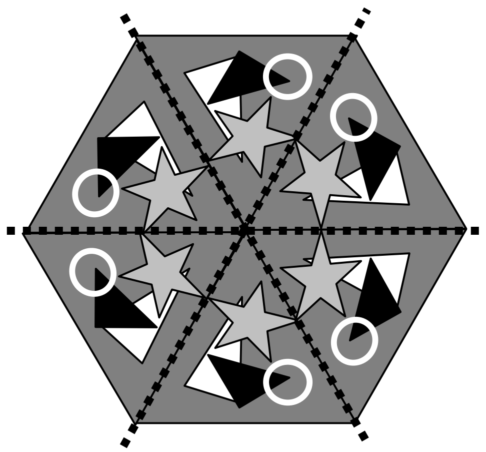
The designs below are called kaleidoscope designs because they are similar to the designs you would see if you looked through a kaleidoscope. In this problem, you will explore the symmetries in these designs. The designs are reproduced on Labsheet 1.3.

A. Look for **reflectional** symmetry in each design. Sketch all the lines of symmetry you find.

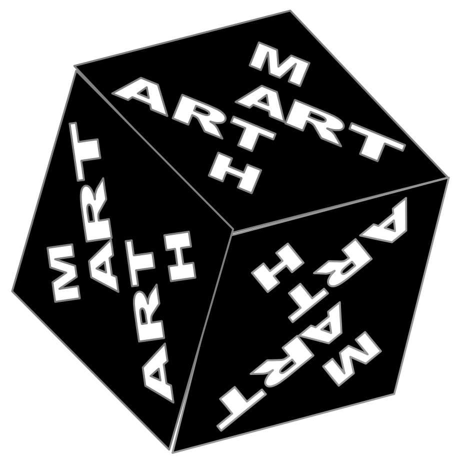
B. Look for **rotational** symmetry in each design. Determine the angle of rotation for each design.
Designs 1, 2, 3 and 6 have a 120° angle of rotation.
Designs 4 and 5 have a 60° angle of rotation.



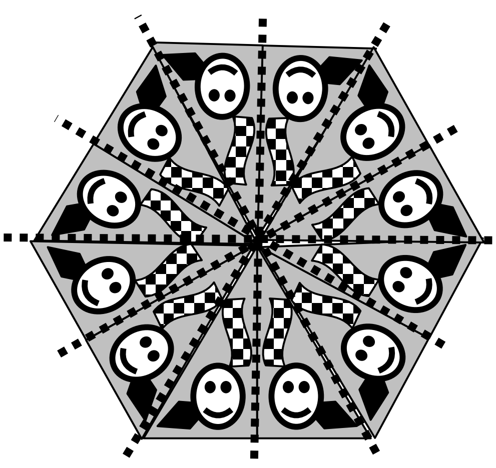
Design 1



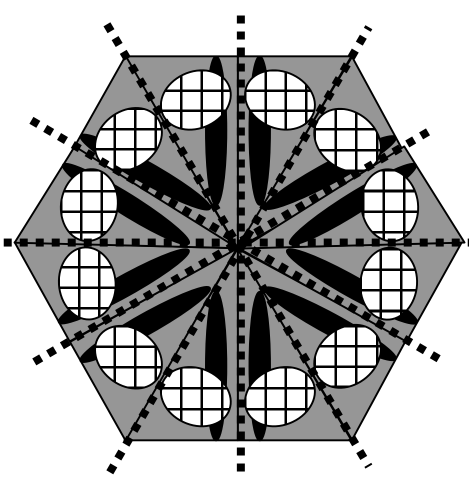
Design 2



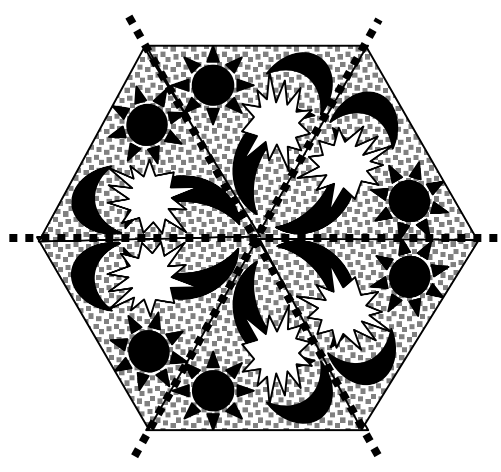
Design 3



Design 4



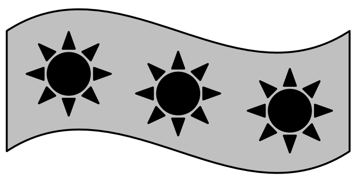
Design 5

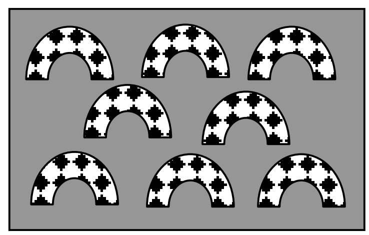


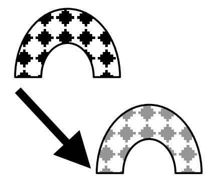
Design 6

Kaleidoscopes, Hubcaps, and Mirrors
Problem 1.4 Notes

A translation is a geometric motion that slides a figure from one position to another. A design has <u>translational symmetry</u> if it can be created by sliding a basic design element in a regular, straight-line pattern. Designs on ribbons, belts, and wallpaper often have translation symmetry.



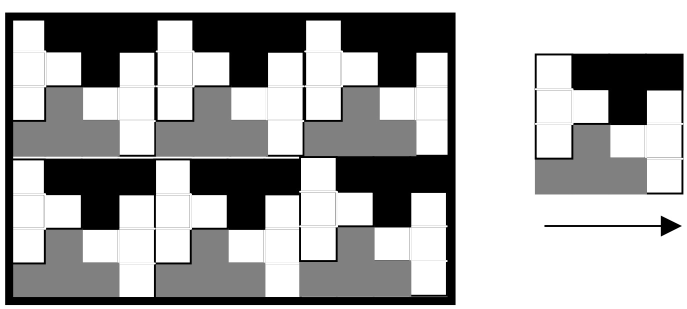


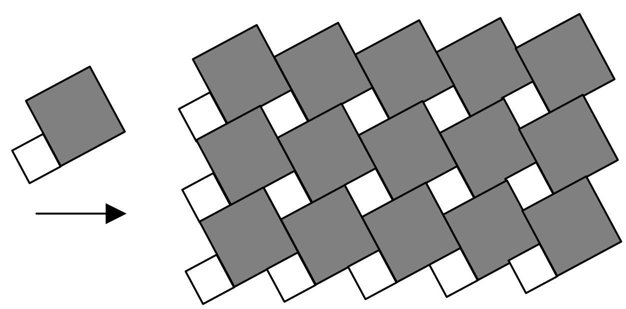


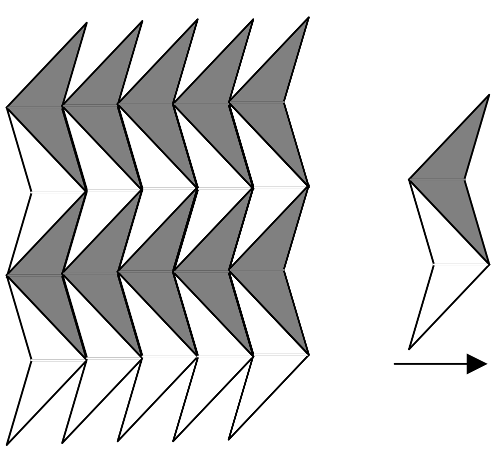
The designs below are tessellations. A **tessellation** is a design made from copies of a single basic design element that cover a surface witout gaps or overlaps. The tessellations are reproduced on Labsheet 1.4. Each tessellation has translation symmetry.

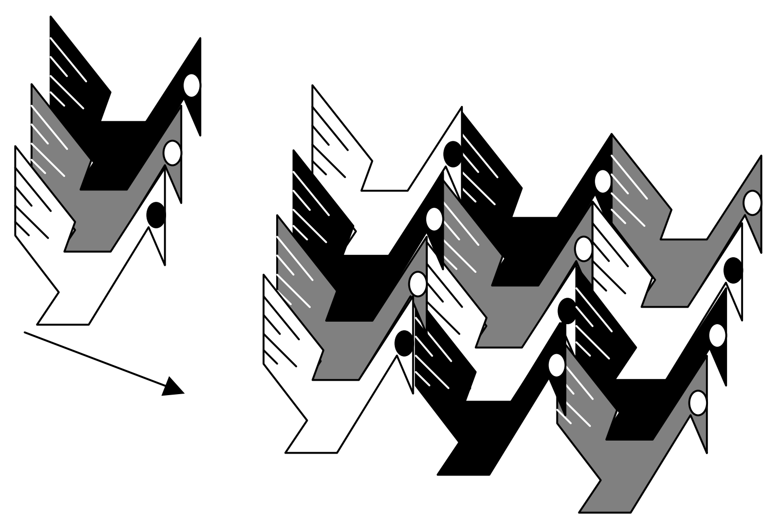
A. Outline a basic design element that could be used to create the tessellation using only translations.

B. Write directions or draw an arrow showing how the basic design element can be copied and slid to produce another part of the pattern.



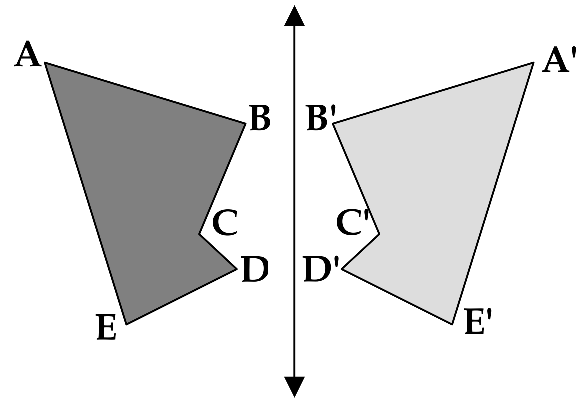






Kaleidoscopes, Hubcaps, and Mirrors
Problem 2.1 Notes

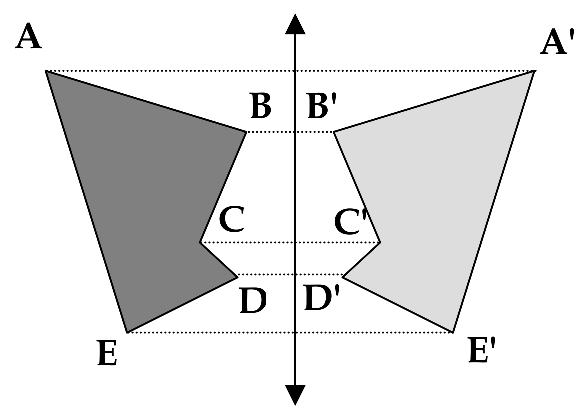
You can draw a figure with reflectional symmetry by finding the mirror image of polygon ABCDE over the line. To help you draw you figure, you could set a mirror on the line to see what the image looks like.



Drawing the mirror image of a figure is an example of a geometric operation called a transformation. **Transformations** produce a copy of a figure in a new positon. The copy is called the *image* of the original figure.

Transformations that create new figures with reflectional symmetry are called **line reflections**. When you make a design using a line reflection it helps to know how the transformation matches each point on a figure to a point on the image of the figure.

A. In the figure above, polygon A'B'C'D'E' is the image of polygon ABCDE under a line reflection. This figure is reproduced on Labsheet 2.1A.



1. Draw a line segment from each vertex of polygon ABCDE to its image on polygon A'B'C'D'E'.

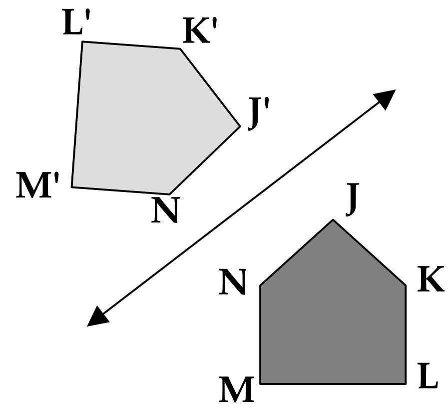
2. Measure the angles formed by each segment you drew and the line of reflection.

3. For each vertex of polygon ABCDE, measure the distance from the vertex of the line of reflection along the segments you drew and the distance from the line of reflection to the image of the vertex.

4. Describe the patterns in your measurements from parts 2 and 3.

   Each vertex and its image are the same distance from the line of reflection. The segment joining each vertex to its image is perpendicular to, or forms a right angle with, the line of reflection.

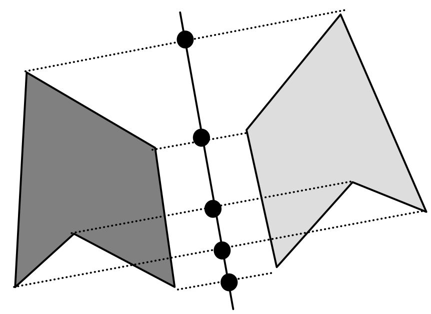
B. The figure shown is reproduced on Labsheet 2.1A. Use what you discovered in part A to draw the image of polgon JKLMN under a reflection over the line. Use only a pencil, a ruler, and an angle ruler or protractor. Desribe how you drew the image.



Draw a line from each vertex of polygon JKLMN to the line of reflection and extending beyond it. Measure the length of the line segment from each vertex to the line of reflection, and measure an equal distance from the line of reflection to the opposite side. Mark the new points J', K', L', M', and N', and connect these new points.

C. The reflection-symmetric design shown is reproduced on Labsheet 2.1A. Use only a pencil, a ruler, and an angle ruler or protractor to find the line of symmetry for this design. Describe how you found the line of symmetry.

Draw line segments joining each pair of corresponding vertices. Measure to find the midopint of each line sgment. The line that joins the midpoints is the line of symmetry.



D. Complete this definition of a line relfection: A line reflection matches each point X on a figure to an image point X' so that... **(1) the distance from each of these paired points to the line of symmetry is the same and (2) the segment XX' is perpendicular to the line of symmetry.**

Follow-Up Problem 2.1

The figures you have reflected in this unit are two-dimensional. Two-dimensional figures are drawn in a plane. You can picture a plane as a sheet of paper that extends infinitely in all directions. When you reflect a figure over a line, you can image that you are reflecting the entire plane over the line.

4. When you perform a line reflection, are any points in the plane in the same location after the reflection? **Only points that line on the line of reflection are in the same location after the line reflection.**

Points that are in the same location after a transformation are called ***fixed points***. Some transformations have fixed points; other transformations move every point in the plane to a new location.

Kaleidoscopes, Hubcaps, and Mirrors
Problem 2.2 Notes

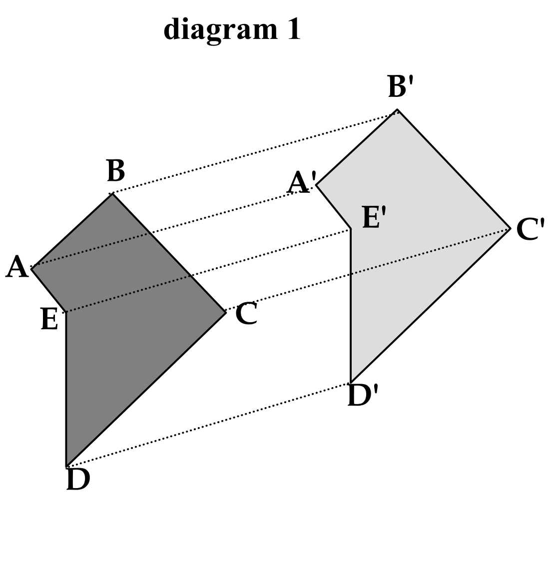
You have seen how line reflections are related to reflectional symmetry. Other types of symmetry also have associated transformation. In Investigation 1, you saw how patterns with translational symmetry could be created by translating, or sliding, a basic pattern. A translation is a type of transformation. In this problem, you will look at examples of translations in search of an answer to this question:

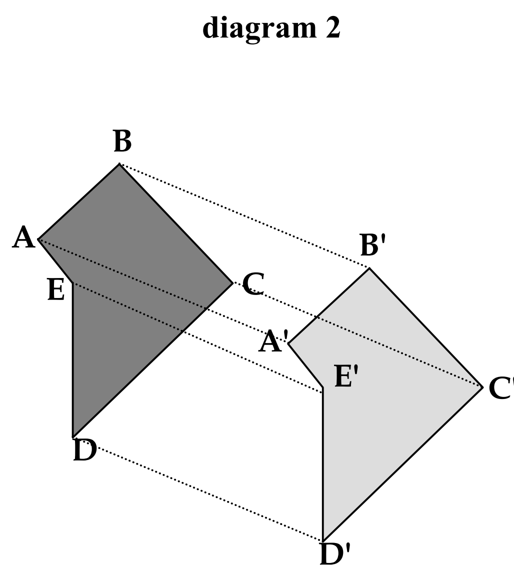
What is the relationship between a figure and its image under a translation?

As you work on each example, think about the instructions you could give so that someone else could re-create the translation exactly.

A. Each diagram below shows polygon ABCDE and its image under a translation. These figures are reproduced on Labsheet 2.2. Do parts 1 and 2 for each diagram.

1. Draw a line segment from each vertex of polygon ABCDE to its image.

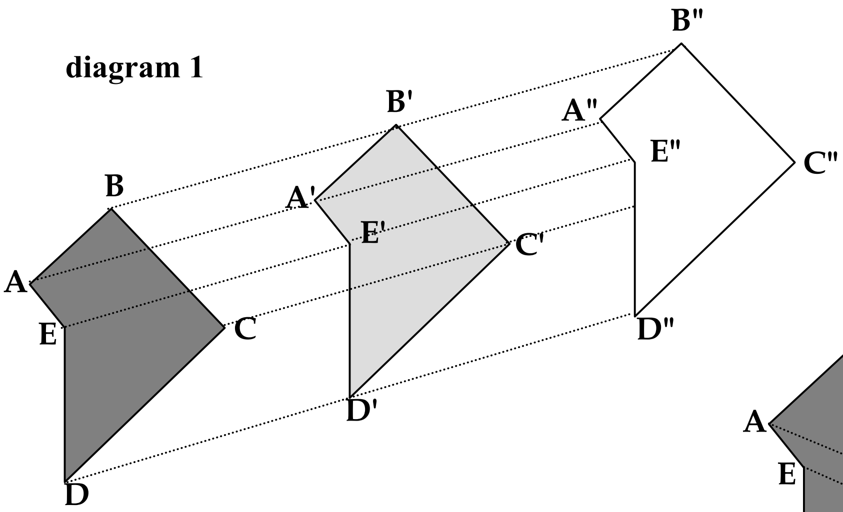


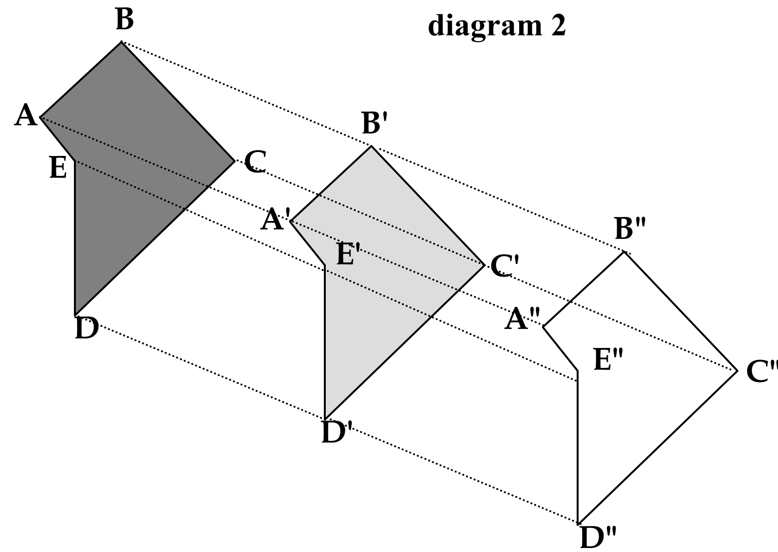


2. Describe the relationship among the line segments you drew.
The connecting line segments are parallel and of the same length.

B. The translations in part A slide polygon ABCDE onto its image, polygon A'B'C'D'E'. Do parts 1-3 for each diagram in part A.

1. By performing the same translation that was used to slide polygon ABCDE to polygon A'B'C'D'E', slide polygon A'B'C'D'E' to create a new image. Label the image A"B"C"D"E".



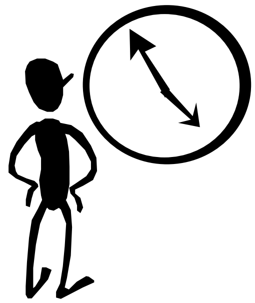


2. Polygon A"B"C"D"E" is the image of polygon ABCDE after two identical translations. How is polygon A"B"C"D"E" related to polygon ABCDE?
Polygon A"B"C"D"E" is twice as far from the original, in the same direction, as polygon A'B'C'D'E' is. The first image's vertices are the midpoints of the line segments connecting an original vertex and its second image.

3. Does your final drawing have translational symmetry?
The final drawing has the beginning of translational symmetry.

C. Complete this definition of a translation: A ***translation*** matches any two points X and Y on a figure to image points X' and Y' so that... **(1) the distance from X to X' is equal to the distance from Y to Y' and (2) the line XX' is parallel to the line YY'.**

Kaleidoscopes, Hubcaps, and Mirrors
Problem 2.3 Notes

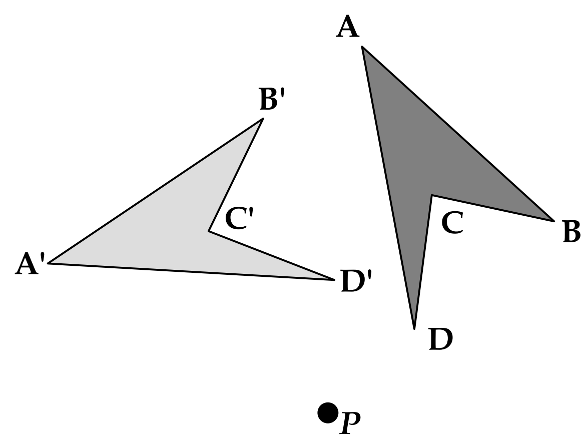


In Investigation 1, you explored rotational symmetry, and you created symmetric designs by rotating figures about a point. A rotation is a type of transformation. The rotations in this unit are *counterclockwise* turns about a point. In this problem, you will try to answer this question:

What is the relationship between a figure and its image under a rotation?

As you work on each example, think about the instructions you could give so that someone else could re-create the rotation exactly.

A. In the figure, polygon A'B'C'D' is the image of polygon ABCD under a rotation of 60°. This figure is reproduced on Labsheet 2.3.



1. What relationship would you expect to find between each vertex, its image, and point P?
If you formed angles by connecting each vertex and its image to the center of rotation, they might measure 60° because the angle of rotation is 60°. Also, each vertex and its image might be the same distance from the center of rotation.

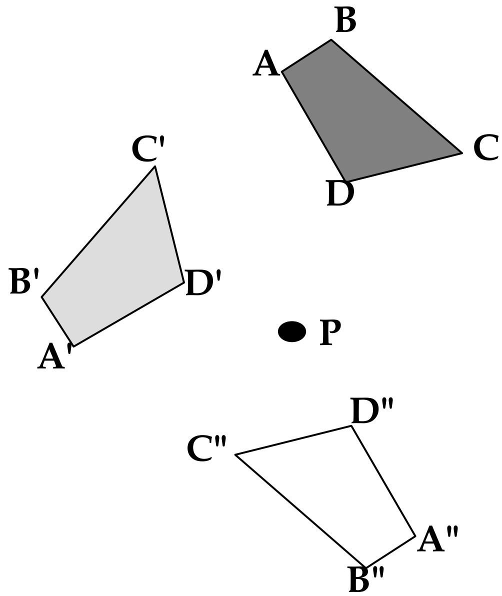
2. For each vertex of polygon ABCD, find the measure of the angle formed by the vertex, point P, and the image of the vertex. For example, find the measure of angle APA'.
All the angles measure 60°.

3. For each vertex of polygon ABCD, find the distance from the vertex to point P and the distance from the image of the vertex to point P. For example, find AP and A'P.
On your labsheet, segments AP & A'P measure 3.8 cm, segment BP & B'P measures 3.4 cm, segments CP & C'P measure 2.75 cm, and segments DP & D'P measure 1.5 cm.

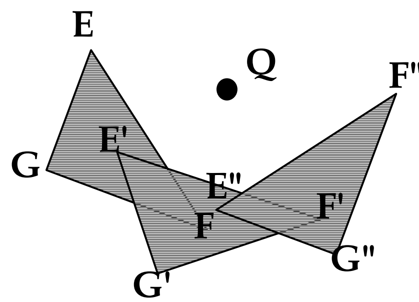
4. What patterns do you see in your measurements? Do these patterns confirm the conjecture you made in part 1?
The angle formed by each vertex, point P, and the image of the vertices is 60°. Each vertex and its image are the same distance from point P.

B. The figures below are reproduced on Labsheet 2.3A. Do parts 1 and 2 for each figure.

1. Perform the indicated rotation and label the image vertices appropriately.



Rotate 90° about point P.



Rotate 90° about point Q.

2. Describe the path each vertex follows under the rotation.
In the quadrilateral, each vertex travels along a circle centered at point P and moves through an angle of 90°. In the triangle, each vertex travels along a circle centered at point Q and travels through an angle of 45°.

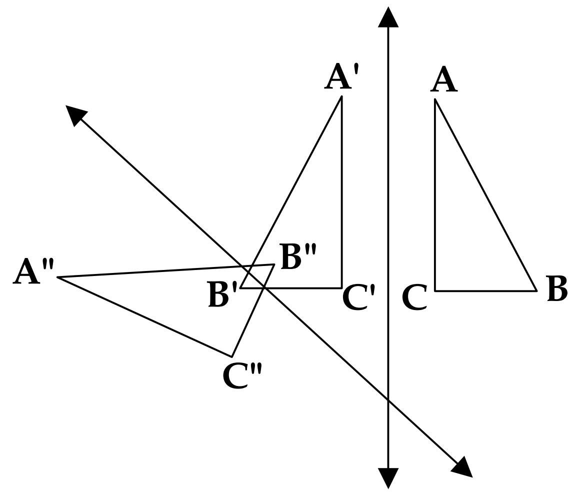
C. For the figures in part B, use the specified rotation to rotate the image of the original polygon. The result is the image of the original polygon after two identical rotations. How does the location of the final image compare with the location of the original polygon?
Each vertex of the final image is the same distance from the center as the original vertex is, and the angle formed has twice the measure of the original angle of rotation.

D. Complete this definition of a rotation: A ***rotation*** of d degrees about a point P match any point X on a figure to an image point X' so that... **the measure of the angle XPX' is d degrees and the distance XP equal the distance X'P.**

Kaleidoscopes, Hubcaps, and Mirrors
Problem 2.4 Notes

Now you will explore what happens when you perform two transformations in a row, the first on the original figure and the second on the image of that figure.

A. 1. The figure is reproduced on Labsheet 2.4A. Reflect triangle ABC over line 1, then reflect the image over line 2. Label the final image A"B"C".



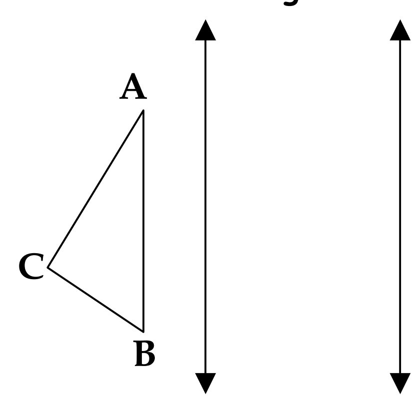
2. For each vertex of triangle ABC, measure the angle formed by the vertex, point I, and the image of the vertex. For example, measure angle AIA". What do you observe.
All the angles measure 78°.

3. For each vertex of triangle ABC, compare the distance from the vertex to point I with the distance from the image of the vertex to point I. What do you observe?
The distances in each pair are equal.

4. Could you move triangle ABC to triangle A"B"C" with a single transformation? If so, describe the transformation.
You could rotate triangle ABC about point I through an angle of 78°.

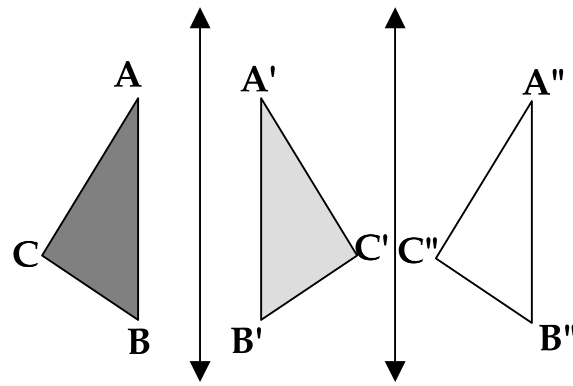
5. Make a conjecture about the result of reflecting a figure over two intersecting lines. Test your conjecture with an example.
Possible conjecture: A reflection of a figure over two intersecting lines is equivalent to a single rotation with the point of intersection as the center of rotation and an angle of rotation that is twice the measure of the angle between the lines.

B. 1. What will happen if you reflect a figure over a line and then reflect the image over a second line that is parallel to the first line? Would the combination of the two reflections be equivalent to a single transformation?
The result might be equivalent to a single translation.



2. Test your conjecture from part 1 on several examples, including the one shown. Do the results support your conjecture? If so, explain why. If not, revise your conjecture to better explain your results.

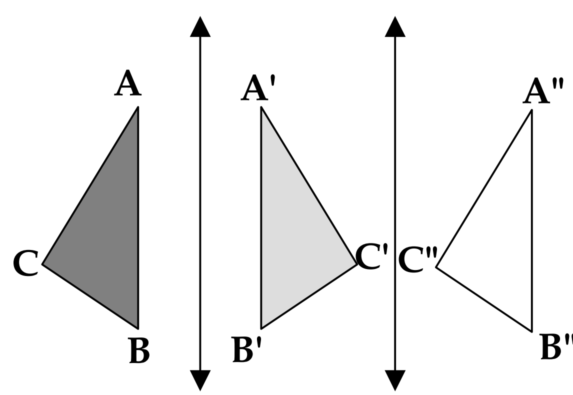
The measures of the line segments between any point and its final image are equal. Because these line segments are perpendicular to the lines of reflection, this transformation can be done as a single translation. The slide must be in a direction perpendicular to the lines of reflection and for a distance of twice the distance between the lines.

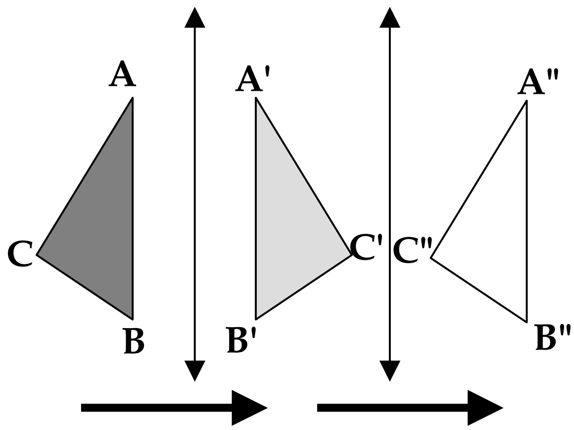


Problem 2.4 Follow-Up

4. What would happen if you translated a figure and then translated the image? Would the combination of translations be equivalent to a single transformation?

Two translations are equivalent to a single translation. The single translation can be found by drawing an arrow to represent the first translation and drawing a second arrow from the end of the first arrow to represent the second translation. A single arrow drawn from the start of the first arrow to the end of the second arrow indicates an equivalent, single translation.





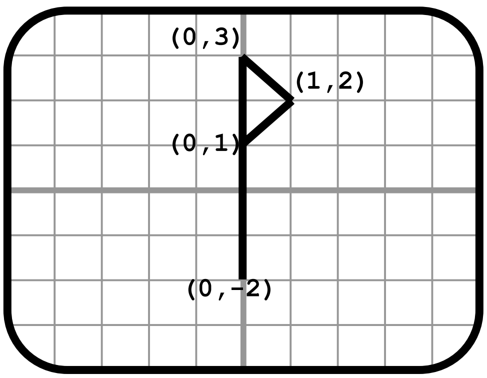
5. What would happen if you rotated a figure about a point and then rotated the image about the same point? Would the combination of rotations be equivalent to a single transformation?

With the same center of rotation, you just add the amount of rotations to find the amount of an equivalent, single rotation.

Kaleidoscopes, Hubcaps, and Mirrors
Problem 3.1 Notes

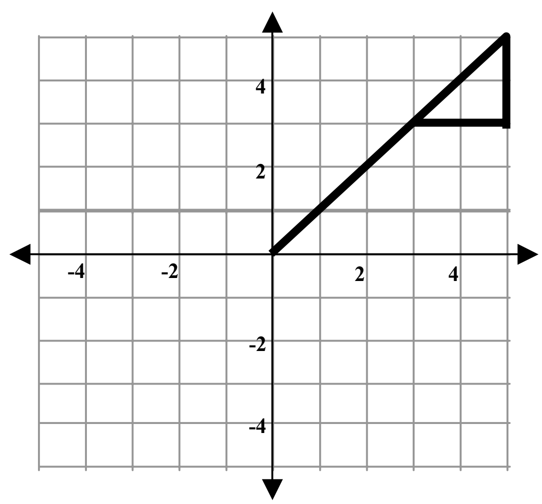
In this investigation, you will explore transformations of figures drawn on coordinate grids. By looking for patterns in your results, you will be able to write some general algebraic rules for transforming a point (x, y) under reflections, translations, and rotations.

The drawing screen in many computer geometry programs is considered to be a coordinate grid. You can create designs by specifying the endpoint of line segments. The flag shown consists of three segments. The commands for creating the flag in a particular geometry program are shown on the next screen. The commands tell the computer to draw segments between the specified endpoints.



Draw:
```
Line [(0, -2), (0, 3)]
Line [(0, 3), (1, 2)]
Line [(1, 2), (0, 1)]
```

A. Suppose you want to re-create the flag below using the geometry program that drew the flag shown above. Complete the commands to create a set of instructions for drawing the flag.



Draw:
```
Line [(0,0), (5,5)]
Line [(3,3), (5,3)]
Line [(5,3), (5,5)]
```

B. Write a set of commands that would draw the image of this flag under a reflection over the **y-axis**.

Draw:
```
Line [(0,0), (-5,5)]
Line [(-3,3), (-5,3)]
Line [(-5,3), (-5,5)]
```

C. Write a set of commands that would draw the image of this flag under a reflection over the **x-axis**.

Draw:
```
Line [(0,0), (5,-5)]
Line [(3,-3), (5,-3)]
Line [(5,-3), (5,-5)]
```

D. Write a set of commands that would draw the image of this flag under a reflection over the line **y=x**.

```
Draw:
Line [(0,0), (5,5)]
Line [(3,3), (3,5)]
Line [(3,5), (5,5)]
```

Problem 3.1 Follow-Up

Use the diagram on pg. 44 of your _Kaleidoscopes, Hubcaps, and Mirrors_ book or Labsheet 3.1B to answer the following questions.

1. List the coordinates of the labeled points.
   A (-3, 3), B (-2, 4), C (3, 3), D (1, 2), E (4, 1), F (2, -2), G (3, -4), H (5, -5), I (-2, -4), J (-4, -1)

2. a. Indicate the images of points A-J under a reflection over the **y-axis**.
   b. List the coordinates of points A'-J'.
   A' (3, 3), B' (2, 4), C' (-3, 3), D' (-1, 2), E' (-4, 1), F' (-2, -2), G' (-3, -4), H' (-5, -5), I' (2, -4), J' (4, -1)

   c. Compare the coordinates of each original point with the coordinates of its image. Use the pattern you see to compete the general rule for finding the image of any points (x, y) under a reflection over the **y-axis**: **(x, y) → (-x, y)**

3. a. Indicate the images of points A-J under a reflection over the **x-axis**.
   b. List the coordinates of pints A"-J".
   A" (-3, -3), B" (-2, -4), C" (3, -3), D" (1, -2), E" (4, -1), F" (2, 2), G" (3, 4), H" (5, 5), I" (-2, 4), J" (-4, 1)

   c. Compare the coordinates of each original point with the coordinates of its image. Use the pattern you see to compete the general rule for finding the image of any points (x, y) under a reflection over the **x-axis**: **(x, y) → (x, -y)**

4. a. Indicate the images of points A-J under a reflection over the **line y=x**.
   b. List the coordinates of pints A'''-J'''.
   A''' (3, -3), B''' (4, -2), C''' (3, 3), D''' (2, 1), E''' (1, 4), F''' (-2, 2), G''' (-4, 3), H''' (-5, 5), I''' (-4, -2), J''' (-1, -4)

   c. Compare the coordinates of each original point with the coordinates of its image. Use the pattern you see to compete the general rule for finding the image of any points (x, y) under a reflection over the **line y=x**: **(x, y) → (y, x)**

Kaleidoscopes, Hubcaps, and Mirrors
Problem 3.2 Notes

The designs below have translational symmetry. In this problem, you will explore the computer commands needed to create these designs.

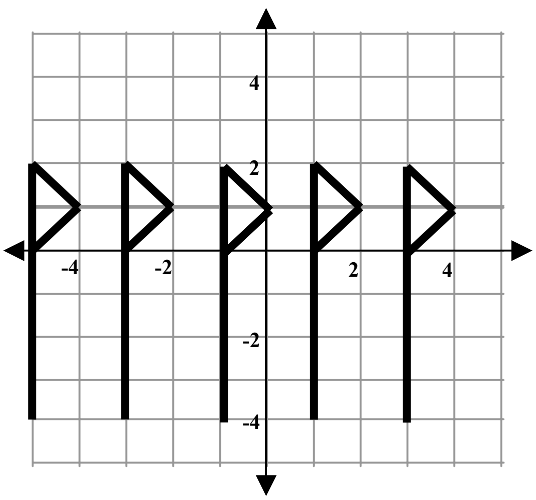


diagram 1

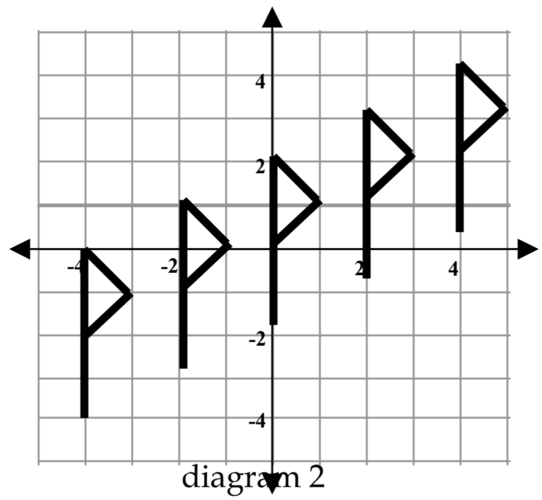


diagram 2

A. 1. In diagram 1, the left most flag can be drawn with these commands:

```
Draw:
Line [(-5,-4), (-5,2)]
Line [(-5,2), (-4,1)]
Line [(-4,1), (-5,0)]
```

These commands draw the vertical segment, then the upper slanted segment, and finally the lower slanted segment. Write sets of commands for drawing the other four flags in diagram 1. Each set of commands should draw the segments in the same order as the commands of the original flag.

flag 2

```
Draw:
Line   [(-3,-4), (-3,2)]
Line   [(-3,2), (-2,1)]
Line   [(-2,1), (-3,0)]
```

flag 3

```
Draw:
Line   [(-1,-4), (-1,2)]
Line   [(-1,2), (0,1)]
Line   [(0,1), (-1,0)]
```

flag 4

```
Draw:
Line   [(1,-4), (1,2)]
Line   [(1,2), (2,1)]
Line   [(2,1), (1,0)]
```

flag 5

```
Draw:
Line   [(3,-4), (3,2)]
Line   [(3,2), (4,1)]
Line   [(4,1), (3,0)]
```

2. Compare the commands for the five flags. Describe a pattern that relates the coordinate of each flag to the coordinates of the flag to its ***right***.

   To find the coordinates of the flag to the right of a given flag, add 2 to each x-coordinate and keep the y-coordinates the same.

3. Describe a pattern that relates the coordinates of each flag to the coordinates of the flag to its ***left***.

   To find the coordinates of the flag to the left of a given flag, subtract 2 to each x-coordinate and keep the y-coordinates the same.

B. 1. Write a set of commands for drawing the left-most flag in diagram 2. Then write comparable instructions for drawing the other four flags.

flag 1
```
Draw:
Line  [(-4,-4), (-4,0)]
Line  [(-4,0), (-3,-1)]
Line  [(-3,-1), (-4,-2)]
```

flag 2
```
Draw:
Line  [(-2,-3), (-2,1)]
Line  [(-2,1), (-1,0)]
Line  [(-1,0), (-2,-1)]
```

flag 3
```
Draw:
Line  [(0,-2), (0,2)]
Line  [(0,2), (1,1)]
Line  [(1,1), (0,0)]
```

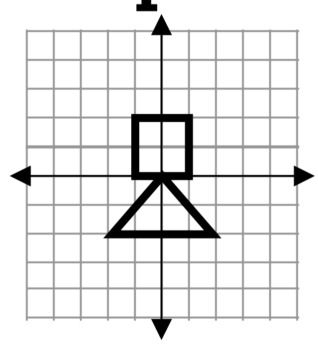
flag 4
```
Draw:
Line  [(2,-1), (2,3)]
Line  [(2,3), (3,2)]
Line  [(3,2), (2,1)]
```

flag 5
```
Draw:
Line  [(4,0), (4,4)]
Line  [(4,4), (5,3)]
Line  [(5,3), (4,2)]
```

2. Compare the commands for the five flags. Describe a pattern that relates the coordinate of each flag to the coordinates of the flag to its ***right***.
To find the coordinates of the flag to the right of a given flag, add 2 to each x-coordinate and 1 to each y-coordinate.

3. Describe a pattern that relates the coordinates of each flag to the coordinates of the flag to its ***left***.
To find the coordinates of the flag to the right of a given flag, subtract 2 to each x-coordinate and 1 from each y-coordinate.

Problem 3.2 Follow-Up



Use the figure to answer questions 1-3.

1. Suppose you want to create a design with translational symmetry by **sliding** this basic figure **along the x-axis**. Consider different designs you could make by using translations of various lengths.

A. In any design created by translating the figure along the x-axis, how are the coordinates of the original figure related to the coordinates of the figure to its right?
The x-coordinate of the figure to the right of a given figure is x plus the length of the translation; the y-coordinate does not change.

B. How are the coordinates of the second copy to the right of the original related to the coordinates of the first copy to the right? How are the coordinates of the second copy related to the coordinates of the original?
The x-coordinate of the figure second to the right of a given figure is the x-coordinate of the first copy to the rights plus the length of the translation. It is also the x-coordinate of the original figure plus twice the amount of the translation. In both cases, the y-coordinate does not change.

C. How are the coordinates of copies to the left of the original related to the coordinates of the original?

The x-coordinate of any figure to the left of a given figure is x minus the length of the translation; the y-coordinate does not change.

2. Consider the designs you could make by sliding this basic figure along the y-axis.

A. In any such design, how are the coordinates of the original figure related to the coordinates of the figure directly above it?

The y-coordinate of the figure directly above a given figure is y plus the length of the translation; the x-coordinate does not change.

B. How are the coordinates of the second copy above the original related to the coordinates of the copy directly below it? How are the coordinates of the second copy related to the coordinates of the original?

The y-coordinate of the second figure above a given figure is the y-coordinate of the first copy plus the length of the translation. It is also the y-coordinate of the original figure plus twice the length of the translations. In both cases, the x-coordinate does not change.

C. How are the coordinates of copies below the original related to the coordinates of the original?

The y-coordinate of any figure below a given figure is y minus the length of the translation; the x-coordinate does not change.

4. Look back at the translations you explored in questions 1-3. In each case, think about the rules for finding the image of any point (x, y). How are the rules similar?

The rules are similar in that something is added to or subtracted from the x-coordinate and something is added to or subtracted from the y-coordinate.

5. Which of these rules describe translations? Explain.

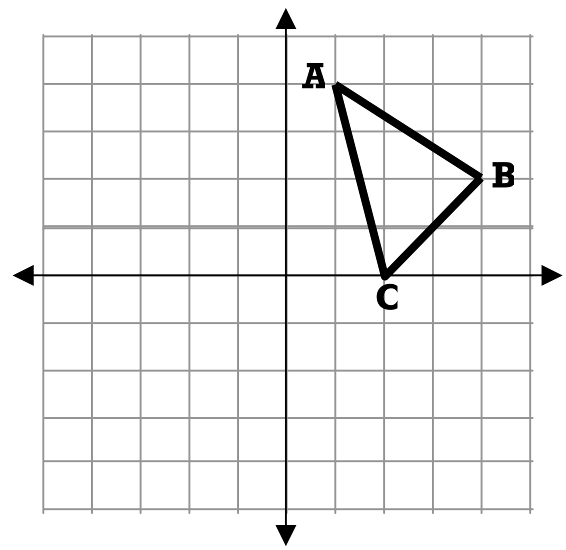
a. $(x, y) \rightarrow (x + 2, y - 3)$
b. $(x, y) \rightarrow (2x, y)$
c. $(x, y) \rightarrow (x +1, 3y)$
d. $(x, y) \rightarrow (x - 2, y + 1)$

The rule $(x, y) \rightarrow (x + 2, y - 3)$ describes a translation that moves each pint 2 units to the right and 3 units down.

The rule $(x, y) \rightarrow (x - 2, y + 1)$ describes a translation that moves each point 2 units to the left and 1 unit up.

Kaleidoscopes, Hubcaps, and Mirrors
Problem 3.3 Notes

You have explored the rules for transforming a point (x, y) to its image under reflections and translations. Write rules for rotations is more difficult. In this problem, you will explore a few simple cases.



A. Copy and complete the commands to create a set of instruction for drawing triangle ABC.

```
Draw:
Line [(-5,-4), (-5,2)]
Line [(-5,2), (-4,1)]
Line [(-4,1), (-5,0)]
```

B. Write a set of commands that would draw the image of triangle ABC under a 90° rotation about the origin.

```
Draw:
Line [(-4,1), (-2,4)]
Line [(-2,4), (0,2)]
Line [(0,2), (-4,1)]
```

C. Write a set of commands that would draw the image of triangle ABC under a 180° rotation about the origin.

```
Draw:
Line [(-1,-4), (-4,-2)]
Line [(-4,2), (-2,0)]
Line [(-2,0), (-1,-4)]
```

D. Write a set of commands that would draw the image of triangle ABC under a 270° rotation about the origin.

```
Draw:
Line [(4,-1), (2,-4)]
Line [(2,-4), (0,-2)]
Line [(0,-2), (4,-1)]
```

E. Write a set of commands that would draw the image of triangle ABC under a 360° rotation about the origin.

```
Draw:
Line [(1,4), (4,2)]
Line [(4,2), (2,0)]
Line [(2,0), (1,4)]
```

Problem 3.3 Follow-Up

1. a. Organize you results from Problem 3.3 to indicate the images of the vertices under each rotation.

90° rotation	180° rotation	270° rotation	360° rotation
A(1, 4) → **(-4, 1)**	A(1, 4) → **(-1, -4)**	A(1, 4) → **(4, -1)**	A(1, 4) → **(1, 4)**
B(4 ,2) → **(-2, 4)**	B(4 ,2) → **(-4, -2)**	B(4 ,2) → **(2, -4)**	B(4 ,2) → **(4, 2)**
C(2, 0) → **(0, 2)**	C(2, 0) → **(-2, 0)**	C(2, 0) → **(0, -2)**	C(2, 0) → **(2, 0)**

b. Describe what happens to the vertices of triangle ABC under each rotation. Be sure to discuss how each rotation affects the coordinates of the points.

- **Under a 90° rotation, the x-and y-coordinates exchange places, and then the sign of the new x-coordinate is reversed.**
- **Under a 180° rotation, the signs for the x- and y-coordinates are reversed. Under a 270° rotation, the x- and y-coordinates exchange places, and then the sign of the new y-coordinate is reversed.**
- **Under a 360° rotation, the x- and y- coordinates are unchanged.**

2. Complete the following statements:

a. A rotation of 90° moves point (x, y) to point **(-y, x)**

b. A rotation of 180° moves point (x, y) to point **(-x, -y)**

c. A rotation of 270° moves point (x, y) to point **(y, -x)**

d. A rotation of 360° moves point (x, y) to point **(x, y)**

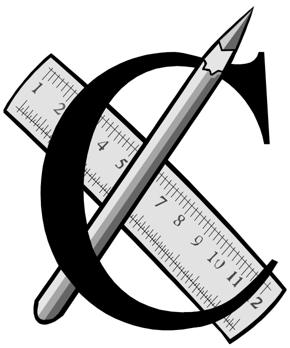
3. What single transformation produces the same final image of triangle ABC as a reflection over the y-axis followed by a reflection over the x-axis? Explain.

A reflection over the y-axis reverses the sign of the x-coordinate and a reflection over the x-axis reverses the sign of the y-coordinate. The two reflections do the following: (x, y) → (-x, y) → (-x, -y). This is the same result as a rotation of 180°

4. What single transformation produces the same final image over triangle ABC as a reflection over the y-axis followed by a reflection over the line y = x? Explain.

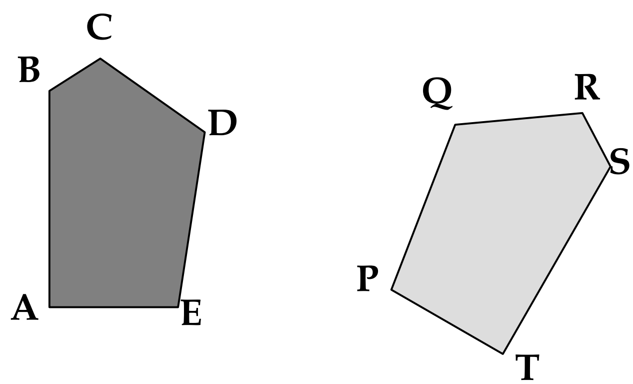
A reflection over the y-axis reverses the sign of the x-coordinate and a reflection over the line y = x reverses the signs of both coordinates. The two reflections do the following: (x, y) → (-x, y) → (-y, x). This is the same result as a rotation of 270°

Kaleidoscopes, Hubcaps, and Mirrors
Problem 3.4 Notes



Congruent figures have the same size and shape. Congruent figures can also be defined using the language of symmetry transformation: *Two figures are congruent if one is an image of the other under a reflection, a translation, a rotation, or some combination of these transformations.* Put more simply, two figures are congruent if you can flip, slide, or turn one figure so that it fits exactly on the other.

Pentagons ABCDE and PQRST are congruent. These same polygons are reproduced on Labsheet 3.4.



A. If you made a copy of one of the pentagons and fit it exactly on the other, which vertices would match?

A and T, B and S, C and R, D and Q, E and P

B. Which pairs of sides in pentagons ABCDE and PQRST are the same lengths?

AB and TS, BC and SR, CD and RQ, DE and QP, EA and PT

C. Which pairs of angles in pentagons ABCDE and PQRST are the same size?

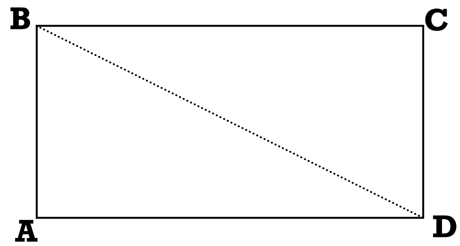
$\angle$A and $\angle$T, $\angle$B and $\angle$S, $\angle$C and $\angle$R, $\angle$D and $\angle$Q, $\angle$E and $\angle$P

D. What combinations of reflections, rotations, and translations would move pentagon ABCDE to fit exactly on pentagon PQRST? Is there more than one possible combination?

Pentagon ABCDE could be translated toward pentagon PQRST until points A and T coincided. Pentagon ABCDE could then be reflected over a line. Other combinations will work.

Problem 3.4 Follow-Up

1. Drawing a diagonal of a rectangle creates two congruent triangles.



a. If you make a copy of triangle ABCD and place it exactly on the other triangle, which vertices will match?

A and C, B and D, D and B

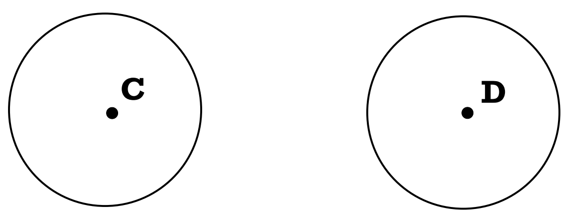
b. What single reflection, rotation, or translation would match one of the triangles exactly with the other?

a rotation of 180° about the midpoint of segment BD

c. Which pairs of sides and angles in the two triangles are congruent?

sides: AB and CD, DA and BC, BD and DB
angles: ∠A and ∠C, ∠ABD and ∠CDB, ∠ADB and ∠CBD

2. a. How could you determine whether these circles are congruent without making copies of them or performing transformations?



You could measure to determine whether their diameters are equal.

b. What transformations could you perform to move one of the circles onto the other to check for congruence?

Possible solutions:

- **translate one circle onto the other along a line joining their centers**
- **rotate one onto the other with the center of rotation being the midpoint of the line segment joining their centers**
- **reflect one onto the other over a line perpendicular to the line segment joining their centers and passing through the midpoint of that segment**

Kaleidoscopes, Hubcaps, and Mirrors
Problem 4.1 Notes

The operations of arithmetic - addition, subtraction, multiplication and division - define ways of putting two numbers together to get a single number. In a similar way, you can think of combining symmetry transformations as an operation that puts two transformations together to produce a single, equivalent transformation.

In this problem, you will explore combinations of symmetry transformations for an equilateral triangle. The notation L_n means a reflection over line n, and the notation R_n means a counterclockwise rotation of n degrees. The symbol $\ast$ represents the combining operation. You can read this symbol as "and the". For example, $L_l \ast L_2 = R_{240}$ means that reflecting the triangle over line l "and then" reflecting over line 2 is equivalent to rotating the triangle 240°.

The table shows the rules of combining symmetry transformations of an equilateral triangle. Each entry is the result of performing the transformation in the left column followed by the transformation in the top row. The two entries already in the table represent the combinations you explored in the introduction:

$$L_1 \ast L_2 = R_{240} \quad \text{and} \quad L_3 \ast R_{120} = L_2$$

$\ast$	R_{360}	R_{120}	R_{240}	L_1	L_2	L_3
R_{360}	R_{360}	R_{120}	R_{240}	L_1	L_2	L_3
R_{120}	R_{120}	R_{240}	R_{360}	L_2	L_3	L_1
R_{240}	R_{240}	R_{360}	R_{120}	L_3	L_1	L_2
L_1	L_1	L_3	L_2	R_{360}	R_{240}	R_{120}
L_2	L_2	L_1	L_3	R_{120}	R_{360}	R_{240}
L_3	L_3	L_2	L_1	R_{240}	R_{120}	R_{360}

Note that transformation R360, a 360° rotation, carries every point back to where it started. As you combine the transformations, you will discover that many combinations are equivalent to R360.

Problem 4.1 Follow-Up

1. Look carefully at the entries in your table. Describe any interesting patterns in the rows, columns, or blocks of entries. What do these patterns tell you about the results of combining rotations and line reflections?

   Each cell contains one of the six symmetry transformations, which means that every combination of two transformations is equivalent to a single transformation. The upper-left and lower-right sections are filled with rotations, and the lower-left and upper-right sections are filled with reflections. Combining a rotation and a reflection is equivalent to a single reflection and combining two rotations or two reflections is equivalent to a single rotation. Every row and every column contains each transformation once and there is a row and a column that contain entries that are identical to the row head and column head.

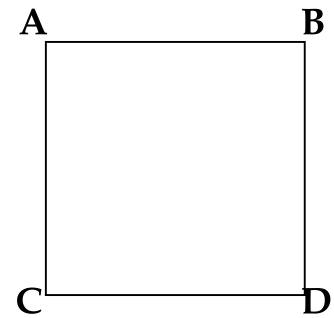
2. Make an operation table for multiplication of the whole numbers 1, 2, 3, 4, 5, and 6. Compare the patterns in your multiplication table with the patterns in your table of transformation combinations. Describe any interesting similarities and differences you discover.

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

In both tables, the entries in the first row match the column heads and the entries in the first column match the row heads. Every entry in the table for the "and then" operation is one of the six transformations that appear in the row and column heads; the entries in the multiplication table consist of more numbers than just those in the row and column heads. The multiplication table has symmetry with respect to the diagonal from the upper-left to the lower-right; that is not true for the "and then" table.

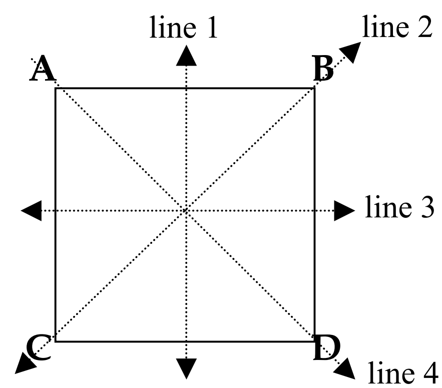
Kaleidoscopes, Hubcaps, and Mirrors
Problem 4.2 Notes

If a polygon has reflectional or rotational symmetry, you can make an operation table to show the results of combining the symmetry transformations for that polygon. There are eight symmetry transformations for a square, including a rotation of 360°. In this problem you will investigate combinations of these symmetry transformations.



A. On Labsheet 4.2B, draw all the lines of symmetry and describe all the rotations that produce images that exactly match the original square. Label the lines of symmetry in clockwise order as line 1, line 2, and so on.

A square has four lines of symmetry and can be rotated 90°, 180°, 270°, or 360° to match the original figure.



B. Cut out a copy of the square, and a copy each vertex label onto the back of the square. Use this copy to explore combinations of symmetry transformations. The operation table is reproduced on labsheet 4.2B. The results of combining pairs of transformations are shown. The rotations are listed first in the column heads, and then the reflections.

*	R_{360}	R_{90}	R_{180}	R_{270}	L_1	L_2	L_3	L_4
R_{360}	R_{360}	R_{90}	R_{180}	R_{270}	L_1	L_2	L_3	L_4
R_{90}	R_{90}	R_{180}	R_{270}	R_{360}	L_2	L_3	L_4	L_1
R_{180}	R_{180}	R_{270}	R_{360}	R_{90}	L_3	L_4	L_1	L_2
R_{270}	R_{270}	R_{360}	R_{90}	R_{180}	L_4	L_1	L_2	L_3
L_1	L_1	L_4	L_3	L_2	R_{360}	R_{270}	R_{180}	R_{90}
L_2	L_2	L_1	L_4	L_3	R_{90}	R_{360}	R_{270}	R_{180}
L_3	L_3	L_2	L_1	L_4	R_{180}	R_{90}	R_{360}	R_{270}
L_4	L_4	L_3	L_2	L_1	R_{270}	R_{180}	R_{90}	R_{360}

Problem 4.2 Follow-Up

1. Compare the patterns in the operation table for the square with the patterns in the operation table for the equilateral triangle. In what ways are the tables similar? In what ways are they different?

Each cell contains one of the possible symmetry transformations, which means that every combination of two transformations is equivalent to a single transformation. The upper-left and lower-right sections are filled with rotations, and the lower-left and upper-right sections are filled with reflections. Combining a rotation and a reflection is equivalent to a single reflection and combining two rotations or two reflections is equivalent to a single rotation. There is a row and column that contain entries that match the row heads and column heads. The tables are different only in that the table for the square contains more symmetry transformations and thus more rows and columns.

2. Make an operation table for multiplication of the whole numbers 1, 2, 3, 4, 5, 6, 7 and 8. Compare the patterns in your multiplication table with the patterns in your table of transformation combinations. Describe any interesting similarities and differences you discover.

x	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	9	12	15	18	21	24
4	4	8	12	16	30	24	28	32
5	5	10	15	20	25	30	35	40
6	6	12	18	24	30	36	42	48
7	7	14	21	28	35	42	49	56
8	8	16	24	32	40	48	56	64

In both tables, the entries in the first row match the column heads and the entries in the first column match the row heads. Every entry in the table for the "and then" operation is one of eight transformations that appear in the row and column heads; the entries in the multiplication table consist of more numbers than just those in the row and column heads. The multiplication table has symmetry with respect to the diagonal from the upper-left to the lower-right; that is not true for the "and then" table.

Kaleidoscopes, Hubcaps, and Mirrors
Problem 4.3 Notes

The operations of addition and multiplication satisfy important properties that are useful for reasoning about expressions and equations.

- The order in which numbers are added or multiplied does not affect the result. This is called the **commutative property**. We say that addition and multiplication are commutative operations. In symbols, if *a* and *b* are real numbers, then

$$a + b = b + a \quad \text{and} \quad a \times b = b \times a$$

- Adding 0 to a number has no effect. Multiplying a number by 1 has no effect. We call 0 and 1 **identity elements**; 0 is the *additive identity*, and 1 is the *multiplicative identity.* In symbols, if *a* is a real number then

$$0 + a = a + 0 = a \quad \text{and} \quad 1 \times a = a \times 1 = a$$

- For any number *a*, the sum of *a* and -*a* is 0, the additive identity. For any nonzero number *a*, the product of *a* and $\frac{1}{a}$ is 1, the multiplicative identity. We call -*a* the *additive inverse* of *a* and $\frac{1}{a}$ the *multiplicative inverse* of *a*. In symbols,

$$a + -a = -a + a = 0 \quad \text{and} \quad a \times \frac{1}{a} = \frac{1}{a} \times a = 1$$

Refer to your operation table for combining symmetry transformations for an equilateral triangle.

A. Is $\ast$ a commutative operation? **The operation $\ast$ is not commutative for an equilateral triangle; order does matter in many cases. For example, reflecting the triangle over line 1 and then over line 1 is equivalent to rotating it 240°; reflecting it over line 2 and then over line 2 is equivalent to rotating it 120°.**

B. Like addition and multiplication, the $\ast$ operation has an identity element. Tell which transformation is the identity element, and explain how you know. **The symmetry transformation R_{360} is an identity element. Rotating the triangle 360° is equivalent to doing nothing because the triangle ends in its original position.**

C. Does each symmetry transformation have an inverse? If so, list each transformation and its inverse. **Each line reflection and the rotation R_{360} can be combined with R_{360} to get the identity element, R_{360}. The remaining reflections, R_{120} and R_{240} are inverse of each other since $R_{120} \ast R_{240}$ -= $R_{240} \ast R_{120} = R_{360}$.**

Problem 4.2 Follow-Up

Refer to your operation table for combining symmetry transformations for a square.

1. Is $\ast$ a commutative operation for the symmetry transformation for a square? **The operation $\ast$ is not commutative for square; order does matter in many cases.**
2. What is the identity element in this situation? **The identity element is R_{360}.**
3. Does each symmetry transformation have an inverse? If so, list each transformation and its inverse. **R_{360}, R_{180}, and each line reflection are inverses of themselves; R_{90} and R_{270} are inverse of each other.**